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Forced Convection in Ducts with a Boundary Condition of the Third Kind

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Introduction

LAMINAR flow of an incompressible fluid in ducts, such as a circular pipe, parallel plates, rectangular ducts, isosceles triangular ducts, and hexagonal ducts is mainly encountered in compact heat exchangers. Numerous investigations have been conducted and the correlations for the friction factor and heat transfer coefficient can be found in handbooks.[1] However, these solutions were obtained for the case of forced convection heat transfer inside the duct subject to the first kind (uniform wall temperature) or second kind (uniform wall heat flux) of boundary conditions, which are the two extreme cases of the third kind when the Biot number approaches infinity or zero. The convection heat transfer from the ambient fluid to the duct can be represented properly by the third kind of thermal boundary condition. In some cases, the third kind of thermal boundary condition also refers to the case where the duct has a finite thermal resistance normal to the wall.[2] Only a few papers have been published on forced convection heat transfer in circular and noncircular ducts with the thermal boundary condition of the third kind. Most of the work focused on the developing region of the circular pipe,[3–5] the flat channel,[6] and the rectangular duct.[7] The forced convection heat transfer of fully developed flow in the circular pipe can be found in numerical work.[8] Convection heat transfer in the noncircular isosceles triangular and hexagonal ducts with the third kind of thermal boundary condition has not as yet been reported. Although the solution for the heat transfer of slug flow in the entrance region of a circular pipe[9] and a rectangular duct[10] has been obtained for the first and second kinds of thermal boundary conditions, the solution for the third kind of thermal boundary condition still needs to be studied. The analytical or numerical solution will be studied for the case of the third kind of thermal boundary condition for both slug and fully developed flows. The slug flow often occurs in fluids with a small Prandtl number, corresponding to the entrance flow.

Analysis

Consideration is given mainly to the steady laminar flow in circular and noncircular ducts, such as flow in parallel plates, the rectangular duct, the isosceles triangular duct, or the hexagonal duct. The fluid flow is assumed as hydrodynamically and thermally fully developed and the thermal properties of the fluid are independent of temperature. In addition, heat generation and viscous dissipation of the fluid are not taken into account. The third kind of boundary condition is imposed for all of the ducts considered here. Therefore, the heat transfer of fluid inside the duct can be described by

$$\rho C_p u \frac{\partial T}{\partial z} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{1}$$

with a boundary condition at the wall

$$-k\frac{\partial T}{\partial n}\bigg|_w = h(T - T_\infty)|_w \tag{2}$$

where ρ, C_p, and k are density, specific heat, and the thermal conductivity of the fluid, respectively; u and T are velocity and temperature; x and y are coordinates in the cross section; $\partial T/\partial z$ is the temperature gradient in the axial direction; h is the heat transfer coefficient of the ambient fluid; and T_∞ is the temperature of the ambient fluid.

After introducing dimensionless variables[8]

$$X = x/l, \qquad Y = y/l, \qquad U = u/u_m, \qquad N = n/l$$
$$\theta = (T - T_\infty)/(T_b - T_\infty), \qquad Bi = h\cdot l/k \tag{3}$$
$$Pe = \rho C_p u_m l/k, \qquad \lambda = -\frac{1}{T_b - T_\infty}\frac{\mathrm{d}(T_b - T_\infty)}{\mathrm{d}z}$$

Eqs. (1) and (2) become

$$\frac{\partial^2\theta}{\partial X^2} + \frac{\partial^2\theta}{\partial Y^2} + \lambda Pe U\theta = 0 \tag{4}$$

$$\frac{\partial\theta}{\partial N}\bigg|_w = -Bi\theta|_w \tag{5}$$

In Eqs. (3–5), Bi is the Biot number, Pe is the Peclet number, and u_m and T_b are the mean velocity and bulk temperature of the fluid, respectively. l is the characteristic length of the duct that is defined in Table 1 for each duct. It is noted that

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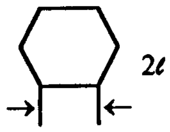
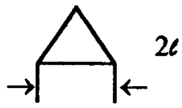
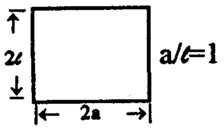
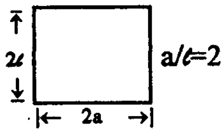
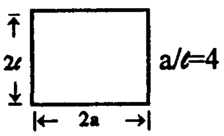
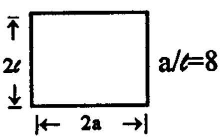
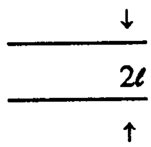
Table 1 Nusselt number results

Duct	Flow type	Biot number 0	0.1	0.2	0.5	1.0	2.0	5.0	10.0	100.0	$+\infty$
	Slug	8.000	7.936	7.872	7.700	7.456	7.098	6.554	6.230	5.834	5.782
	Developed	4.36	4.33	4.28	4.22	4.12	4.04	3.84	3.76	3.66	3.66
	Slug	7.17	7.11	7.01	6.76	6.47	6.12	5.74	5.57	5.39	5.37
	Developed	3.86	3.81	3.77	3.67	3.58	3.49	3.41	3.37	3.34	3.34
	Slug	4.02	4.03	4.05	4.11	4.19	4.30	4.42	4.44	4.40	4.25
	Developed	1.89	1.91	1.93	1.99	2.07	2.17	2.31	2.39	2.47	2.47
	Slug	6.000	5.961	5.925	5.827	5.697	5.524	5.272	5.131	4.957	4.935
	Developed	3.09	3.07	3.06	3.04	3.02	2.99	2.98	2.98	2.98	2.98
	Slug	6.000	5.975	5.954	5.903	5.846	5.774	5.666	5.595	5.497	5.483
	Developed	3.02	3.03	3.05	3.08	3.14	3.21	3.31	3.35	3.39	3.39
	Slug	6.000	6.080	6.157	6.352	6.559	6.734	6.811	6.791	6.723	6.710
	Developed	2.93	3.06	3.15	3.44	3.73	4.00	4.25	4.34	4.43	4.44
	Slug	6.003	6.346	6.642	7.252	7.734	8.038	8.115	8.062	7.940	7.919
	Slug	12.00	11.92	11.85	11.65	11.40	11.04	10.55	10.26	9.915	9.867
	Developed	8.24	8.20	8.17	8.09	8.00	7.88	7.73	7.65	7.55	7.54

λPe is related to the boundary condition of the third kind. Therefore, based on the energy balance, λPe can be expressed as

$$\lambda Pe = \int_S Bi\theta \frac{dS}{A} \tag{6}$$

where A is the dimensionless cross section area and S is the dimensionless perimeter of the duct. The integral in Eq. (6) is performed around the boundary of the duct. Furthermore, it can be seen from Eq. (6) that λPe is always positive. Thus, let

$$\beta^2 = \lambda Pe \tag{7}$$

Eq. (4) is then written as

$$\frac{\partial^2\theta}{\partial X^2} + \frac{\partial^2\theta}{\partial Y^2} + \beta^2 U\theta = 0 \tag{8}$$

It is worthwhile to point out that Eq. (6) establishes the explicit function of λPe with the Bi number, which makes it possible to obtain the analytical solutions of Eq. (8) in the case of slug flow in circular pipes, parallel plates, and rectangular ducts, and makes it simple to obtain the numerical solution of Eq. (8) in the case of fully developed flow because λ is no longer treated numerically. This is an improvement over the method used by Sparrow and Patankar.[8] Once the temperature field is obtained, the Nusselt number inside the duct will be calculated by

$$Nu = (d_h/l)/[(S/\beta^2 A) - (1/Bi)] \tag{9}$$

The previous equations are written in Cartesian coordinates. They are solved for all noncircular ducts. For the circular pipe, the cylindrical coordinates are used. The corresponding dimensionless energy equation and its boundary condition can be written as

$$\frac{d^2\theta}{d\eta^2} + \frac{1}{\eta}\frac{d\theta}{d\eta} + \beta^2 U\theta = 0 \tag{10}$$

$$\left.\frac{d\theta}{d\eta}\right|_{\eta=1} = -Bi\theta|_{\eta=1} \tag{11}$$

Table 2 Analytical solutions for slug flow

	Dimensionless temperature, θ	Eigenvalue, β	Dimensionless coordinates
Circular pipe	$\theta = \beta J_0(\beta\eta)/[2J_0'(\beta)]$	$\beta = BiJ_0(\beta)J_0'(\beta)$	$\eta = r/R$
Parallel plates	$\theta = \beta \cos(\beta X)/\sin \beta$	$\beta = Bi/\tan \beta$	$X = x/l$
Rectangular duct	$\theta = \beta_1\beta_2 \cos(\beta_1 X \cdot l/a)$ $\cos(\beta_2 Y)/(\sin \beta_1 \cdot \sin \beta_2)$	$\beta_1 = Bi \cdot c_0 \tan \beta_1 \cdot a/l$ $\beta_2 = Bi \cdot c_0 \tan \beta_2$ $\beta^2 = \beta_1^2 \cdot (l/a)^2 + \beta_2^2$	$X = x/l,\ Y = y/l$

where $\eta = r/R$. R is the radius of the pipe. Equations (6), (7), and (9) are still applicable.

Results and Discussion

Analytical Solutions for Slug Flow

One of the objectives of the present study is to obtain an analytical solution for slug flow. This is done by solving Eqs. (8) or (10) with the standard method of separation of variables. Solutions of the temperature profile of the fluid for slug flow in the circular pipe, parallel plates, and rectangular ducts are listed in Table 2. The solutions of the eigenvalue equations in Table 2 can be solved numerically or found in the literature.[11] However, the solution for the isosceles triangular and hexagonal ducts is obtained using the numerical method of fully developed flow, which is described in the following section.

Numerical Solutions for Fully Developed Flow

In the case of fully developed flow, the energy equation, Eqs. (8) or (10), was solved by the control volume finite difference method. For the purpose of choosing a grid size, the momentum equation for fully developed flow in the duct is numerically solved first and then the velocity profile and the friction factor are compared with the data available in the literature (the errors of friction factor are less than 0.1%, whenever comparison is available[1]). Afterwards, the velocity profile is introduced and the same grid is used to solve the energy equation. The dimensionless temperature profiles of fully developed flow in the circular pipe, the rectangular duct, the isosceles triangular duct, the parallel plates, and the hexagonal duct are obtained and the Nusselt number is calculated by Eq. (9). The $Bi \to 0$ and $Bi \to \infty$ are assigned to 10^{-3} and 10^{+30} in the numerical computation. The grid independence is also tested by solving the slug flow heat transfer and the fully developed flow of the first and second kind of boundary conditions, and the solutions are compared with some of the analytical results in Table 1 for slug flow and the results in the literature[1] for fully developed flow of the first and second kind of boundary conditions. The Nusselt number values of our results for all the comparable cases are the same as the corresponding values from the analytical solution or from the literature, at least for the first three digits.

Nusselt Numbers Results

The Nusselt number is calculated from Eq. (9) for all cases. The Nusselt numbers for slug flow and fully developed flow are tabulated in Table 1. The Biot number varies from 0 to ∞, which refers to the two extreme limitations of constant heat flux and constant wall temperature on the boundary of the duct. It is found that the Nusselt numbers do change as the Biot number changes from a smaller to a larger value in both the slug flow and the fully developed flow. However, the Biot number effect on the Nusselt number is different for different ducts. As the Biot number increases, the Nusselt numbers for the rectangular ducts and the isosceles triangular ducts always increase, while the Nusselt numbers for the circular pipes, the hexagonal ducts, and the parallel plates decrease. The greater changes in the Nusselt number of the rectangular duct and the isosceles triangular duct are observed in slug flow, as well as fully developed flow. Using the first or second kind of boundary condition to approximate the convective boundary condition may cause up to 20% error. This is probably due to the sharp corners in the isosceles triangular and rectangular ducts. In these ducts, forced convection heat transfer will be enhanced when the heat transfer of the ambient fluid to the duct is stronger.

Conclusions

Heat transfer in circular and noncircular ducts has been investigated analytically and numerically in the case of slug flow and fully developed flow. The boundary condition of the third kind is applied to the boundary of the duct. The Nusselt numbers have been calculated. It was found that the Biot number has a significant effect on the Nusselt number for some cases, such as rectangular ducts. An attempt of simply using the first or second kinds of boundary conditions may result in up to 20% error in predicting the Nusselt number. Furthermore, the Nusselt number of the slug flow and fully developed flow in these ducts (Table 1) provides basic information for the design of heat exchangers.

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